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Conditionally solvable path integral problems

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Abstract. Some specific conditionally exactly solvable potentials are discussed within the path integral formalism. They generalize the usually known potentials by the incorporation of fractional power behaviour and strongly anharmonic terms. We find four different kinds of such potential: the first is related to the Coulomb potential, the second is an anharmonic confinement potential, and the third and fourth are related to the Manning–Rosen potential.

1. Introduction

In recent years there has been enormous success in solving path integrals exactly. Milestones in the development have been the path integral solutions by Feynman of the harmonic oscillator [1, 2], the path integral solution of the radial harmonic oscillator [3–6], and the path integral solution of the Pöschl–Teller and modified Pöschl–Teller potentials [7–9], respectively. All these kinds of problem have in common that they correspond to either a Gaussian, Besselian or Legendrian path integration. The couplings and parameters of the potentials are always assumed to take on arbitrary real values. An extensive list of all potential problems along with other path integral solutions, say in homogeneous spaces [10], will appear soon in a '*Table of Feynman Path Integrals*' [11]; a classification scheme has already been announced in [12, 13]. It is remarkable that almost all of these solutions can be understood in terms of a group path integration, be it a group path integration on the entire group space, or where an extension with the introduction of additional dummy variables is necessary [9, 14].

However, there are a couple of problems which generalize the well known problems which are not entirely soluble by their own accord in the sense that all possible parameters can be freely chosen. These considerations can be made, of course, within the Schrödinger equation approach or within the path integral formalism. In the following I will only be concerned with the path integral approach. One set of such problems is called 'quasi-exactly solvable' [15]. This means that a certain constraint on the parameters must be imposed, and then only a few low-lying energy levels together with the wavefunctions can be stated. Another set of problems is called 'conditionally exactly solvable' [16–19]. They modify the usual potentials in quantum mechanics in a specific way such that they are quantum mechanically exactly solvable; however, the parameters and the coupling of the potentials are not completely free to choose.

In this article I discuss four different potentials of the latter kind. The first potential generalizes and modifies the Coulomb potential. A $1/\sqrt{r}$ term is incorporated and adds significant long-range behaviour to the usual Coulomb interaction. As it turns out a

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specific form of the centrifugal barrier must be chosen (actually attractive) in order that the corresponding path integral can be solved. As has been pointed out by Stillinger [20] this particular set-up can produce a wide barrier around an attractive origin. By manipulating the parameters the ground-state energy can be raised and lowered, and can also be moved to zero-energy; hence a potential with a bound-state character and a resonance character can be studied at the same time. Artificial as this looks, there exists an electrostatic charge density which actually produces such a potential field. A point charge located at the origin is accompanied by strong $1/r^4$ and $1/r^{5/2}$ behaviour [20].

The second potential is a confinement potential with a dominant $r^{2/3}$ behaviour for $r \to \infty$. Again a specific form of the radial potential (attractive) is required in order that the path integral can be solved. The potential is therefore an anharmonic radial oscillator. It has played a role in the modelling of quark–antiquark forces for mesons in nuclear physics [21].

As it turns out, both potentials are rather complicated concerning the proper formulation of the quantization condition. In each case it is necessary to solve a transcendental equation involving a parabolic cylinder function. This very point has been ignored in [16, 19], where a naive solution was claimed. The authors did not take into account that the radial problem remains a radial problem even after the transformations, and the deceiving regularity of the transformed problem (a shifted harmonic oscillator) does not allow a coordinate continuation to the entire $\mathbb{R}$ (I sketch the naive solution, though). As we will see, it is not possible to state the propagator exactly. However, the corresponding energy-dependent Green function can be stated in closed form. The proper quantization conditions follow from the poles of the Green functions.

The third and fourth potentials are modifications of a Eckart potential [22] or a Rosen–Morse oscillator [23] and model potential troughs. These kinds of potential play a role in the theory of molecules [22, 23], solitons and reflectionless potentials [24]. In the first of the two potentials the proper transformed potential is of the Manning–Rosen type [25], which was also used as a screened Coulomb potential [26, 27] with exponential decay (s-wave Yukawa potential). The second leads to a hyperbolic Scarf-like potential [28, 29] with an even stronger screening of the potential energy. The connection to supersymmetric quantum mechanics with all these potentials has been pointed out by Dutt *et al* [18], Nag *et al* [19] and Papp [30].

Although exactly solvable, these potentials are complicated enough to be of serious consideration in modelling actual physical forces. By choosing a path integral approach we succeed in gaining comprehensive information about the bound-state solutions of these potentials (if they exist), and what is often more important, in the scattering states which eventually allow for the calculation of cross sections and phase shifts which will be taken into consideration elsewhere.

This article is organized as follows. In the following part I sketch some necessary information concerning transformation techniques in the path integral. In the third section I present the four 'conditionally exactly solvable' potentials labelled V_1, V_2, V_3 and V_4. The well-established spacetime transformation technique reduces each path integral problem to an already known one. The final result in each case includes the statement of the corresponding Green function. For the first two potentials, where the bound-state solutions are only implicitly known through a transcendental equation, this is sufficient. In the other two the bound-state wavefunctions, the energy spectrum, and the continuous states are displayed explicitly. The last section contains a summary and a short discussion.

2. Spacetime transformation technique

In order to make the article self-contained, let us cite shortly the spacetime transformation technique, e.g., [5, 11–13, 31–39], and references therein. We consider a path integral

$$K(x'', x'; T) = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]\mathrm{d}t\right\} \tag{1}$$

where it is assumed that the potential $V(x)$ is so complicated that a direct path integral evaluation is not possible. Now a transformation $x = F(q, t)$ and the so-called 'time transformation' is implemented by introducing a new 'pseudo-time' s''. In order to do this, one first makes use of the operator identity

$$\frac{1}{H - E} = f_r(x, t)\frac{1}{f_l(x, t)(H - E)f_r(x, t)}f_l(x, t) \tag{2}$$

where H is the Hamiltonian corresponding to the path integral $K(T)$, and $f_{l,r}(x, t)$ are functions in x and t, multiplying from the left or from the right, respectively, onto the operator $(H-E)$. Secondly, the introduced pseudo-time s'' is assumed to obey the constraint

$$\int_0^{s''} \mathrm{d}s f_l(F(q(s), s)) f_r(F(q(s), s)) = T = t'' - t' \tag{3}$$

and has, for all admissible paths, a unique solution $s'' > 0$ given by

$$s'' = \int_{t'}^{t''} \frac{\mathrm{d}t}{f_l(x, t) f_r(x, t)} = \int_{t'}^{t''} \frac{\mathrm{d}s}{F'^2(q(s), s)}. \tag{4}$$

Here one has made the choice $f_l(F(q(s), s)) = f_r(F(q(s), s)) = F'(q(s), s)$ in order that in the final result the metric coefficient in the kinetic energy term is equal to one. A convenient way to derive the corresponding transformation formulae uses the energy-dependent Green function $G(E)$ of the kernel $K(T)$ defined by

$$G(q'', q'; E) = \left\langle q''\left|\frac{1}{H - E - \mathrm{i}\epsilon}\right|q'\right\rangle = \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}(E+\mathrm{i}\epsilon)T/\hbar} K(q'', q'; T) \tag{5}$$

where a small positive imaginary part ($\epsilon > 0$) has been added to the energy E. (Usually we do not explicitly write the $\mathrm{i}\epsilon$, but will tacitly assume that the various expressions are regularized according to this rule.) For the path integral $K(T)$ one obtains the following transformation formulae ($F(q, t) \equiv F(q)$ time-independent case only)

$$K(x'', x'; T) = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G(q'', q'; E) \tag{6}$$

$$G(q'', q'; E) = \frac{\mathrm{i}}{\hbar}[F'(q'')F'(q')]^{1/2} \int_0^\infty \mathrm{d}s'' \hat{K}(q'', q'; s'') \tag{7}$$

with the transformed path integral $\hat{K}$ given by

$$\hat{K}(q'', q'; s'') = \int\limits_{q(0)=q'}^{q(s'')=q''} \mathcal{D}q(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{q}^2 - F'^2(q)(V(F(q) - E) - \Delta V(q)\right]\mathrm{d}s\right\} \tag{8}$$

and the quantum potential ΔV has the form

$$\Delta V(q) = \frac{\hbar^2}{8m}\left(3\frac{F''^2}{F'^2} - 2\frac{F'''}{F'}\right). \tag{9}$$

These formulae are sufficient for our purposes.

3. The potentials

3.1. The modified Coulomb potential

The first potential we are going to study has the following form ($r > 0$):

$$V_1(r) = \frac{\hbar^2}{2m}\frac{\gamma}{r^2} - \frac{Zq^2}{r} + \frac{b}{\sqrt{r}}. \tag{10}$$

Zq^2 is a Coulomb coupling, $b \in \mathbb{R}$, and γ is a constant which will be determined. Obviously, this potential is a generalization of a pure Coulomb potential. The centrifugal term usually makes no difficulty, but the $r^{-1/2}$ long-range term significantly alters the behaviour of the potential for $r \to \infty$. I proceed in the canonical manner and perform a coordinate transformation $r = u^2, \dot{r} = 2u\dot{u}$ together with a time transformation $\Delta t_j = 4u_j u_{j-1}\Delta s_j$ in each short-time interval Δt_j to the new time $s(t) = \int_{t'}^{t} \mathrm{d}s/u^2(s)$. I obtain

$$\begin{aligned}K^{(V_1)}(r'', r'; T) &= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \left(\frac{\hbar^2}{2m}\frac{\gamma}{r^2} - \frac{Zq^2}{r} + \frac{b}{\sqrt{r}}\right)\right]\mathrm{d}t\right\}\\ &= 2(r'r'')^{1/4}\int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\mathrm{e}^{4\mathrm{i}Zq^2s''/\hbar}\\ &\quad\times \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{u}^2 - \left(\frac{\hbar^2}{2m}\frac{16\gamma+3}{4u^2} - 4Eu^2 + 4bu\right)\right]\mathrm{d}s\right\}.\end{aligned} \tag{11}$$

This path integral as it stands is not solvable. However, if we set $\gamma = -\frac{3}{16}$, the centrifugal term vanishes due to $\Delta V = 3\hbar^2/8mu^2$, and we seem to have a shifted harmonic oscillator path integral. Let us for the moment continue with the naive analysis according to [16, 19], where $u \in \mathbb{R}$ is assumed. Performing the additional variable shift $v = u - b/2E$ we get

$$\begin{aligned}K^{(V_1)}(r'', r'; T) &= 2(r'r'')^{1/4}\int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'\mathrm{e}^{\mathrm{i}s''(4Zq^2-b^2/E)/\hbar}\\ &\quad\times \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_0^{s''}(\dot{v}^2 - \omega^2 v^2)\,\mathrm{d}s\right]\end{aligned} \tag{12}$$

where $\omega^2 = -8E/m$. Provided $v \in \mathbb{R}$, the last expression can now be analysed in various ways. In order to obtain the discrete spectrum one can insert the explicit form of the propagator of the harmonic oscillator and can expand it by means of the Mehler formula to obtain the wavefunctions and the energy spectrum of the bound states of the path integral (11); second, one can insert the explicit form of the propagator and use a dispersion relation to obtain the wavefunctions of the continuous spectrum. The third possibility is to insert the Green function of the harmonic oscillator: by analysing its analytic pole structure it follows that the Green function of the harmonic oscillator must be evaluated at the energy $E \to 4Zq^2 - b^2/E$, i.e. by performing the s''-integration we obtain the Green function of the path integral (12)

$$G(v'', v'; E) = \sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma(-\nu)\,D_\nu\!\left(\sqrt{\frac{2m\omega}{\hbar}}\,v_>\right)D_\nu\!\left(-\sqrt{\frac{2m\omega}{\hbar}}\,v_<\right). \tag{13}$$

Here $\nu = -\frac{1}{2} + (4Zq^2 - b^2/E)/\hbar\omega$, and $D_\nu(z)$ is a parabolic cylinder function [40, p 1064]. The poles of $G(E)$ yield the bound-state energy spectrum and wavefunctions and the cut the continuous spectrum. The poles are determined by the poles of the Γ-function and we obtain the 'quantization condition'

$$4Zq^2 - b^2/E_n = 2\hbar(n+\tfrac{1}{2})\sqrt{-2E_n/m} \qquad n \in \mathbb{N}_0. \tag{14}$$

With some algebraic manipulations this can be cast into a cubic equation in terms of E_n yielding

$$(n+\tfrac{1}{2})^2\hbar^2 E_n^3 + 2mZ^2q^4E_n^2 - Zq^2b^2mE_n + mb^4/8 = 0. \tag{15}$$

From the generally three solutions of this equation, the physically relevant one is selected by requiring that for $b = 0$ a Coulombic spectrum should emerge. Hence we get

$$E_n = \sqrt[3]{\sqrt{D} - \frac{Q}{2}} - \sqrt[3]{\sqrt{D} + \frac{Q}{2}} - \frac{R}{3} \tag{16}$$

$$D = \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2 \qquad P = \frac{3S - R^2}{3} \qquad Q = \frac{2R^3}{27} - \frac{RS}{3} + T \tag{17}$$

$$R = \frac{2mZ^2q^4}{\hbar^2(n+\frac{1}{2})^2} \qquad S = -\frac{mZq^2b^2}{\hbar^2(n+\frac{1}{2})^2} \qquad T = \frac{mb^4}{8\hbar^2(n+\frac{1}{2})^2}. \tag{18}$$

The bound-state wavefunctions then have the form

$$\Psi_n^{(V_1)}(r) = N_{E_n}\left(\sqrt{\frac{m\omega_n}{\pi\hbar}}r\frac{1}{2^n n!}\right)^{1/2}$$
$$\times H_n\left[\sqrt{\frac{m\omega_n}{\hbar}}\left(\sqrt{r} - \frac{b}{2E_n}\right)\right]\exp\left[-\frac{m\omega_n}{2\hbar}\left(\sqrt{r} - \frac{b}{2E_n}\right)^2\right] \tag{19}$$

$$N_{E_n} = \left(\frac{2|E_n|^3}{(n+\frac{1}{2})[3(E_n + R/3)^2 - R^2/3 + S]}\right)^{1/2}. \tag{20}$$

$H_n(x)$ are Hermite polynomials [40, p 1033]. Up to the additional proper normalization factor N_{E_n} this is the result of [16, 19]. Let us note that for the case $Z = 0$ we can repeat the analysis and obtain a discrete energy spectrum according to

$$E_n = -\sqrt[3]{\frac{mb^4}{8\hbar^2(n+\frac{1}{2})^2}}. \tag{21}$$

This energy spectrum is of genuine non-Coulombic behaviour.

However, this easy-to-obtain solution cannot be considered as correct! As pointed out in [20] a wavefunction with $r^{1/4}$ behaviour at the origin of a singular potential is physically unacceptable [41]. Therefore we must discard the above solution of [16, 19] entirely. In (12) we have made the implicit assumption that it is possible to extend the variable v to the entire $\mathbb{R}$. This is in contrast to the one-dimensional Kustaanheimo–Stiefel transformation which maps $\mathbb{R}^+ \to \mathbb{R}^+$ [42, 43]. In the usual radial Coulomb problem this is an obvious mapping because the radial Coulomb problem in the variable r is mapped onto a radial harmonic oscillator in the variable u. Therefore the path integral (11) is a radial path integral with $u > 0$, and the path integral (12) is a radial path integral with $v > -b/2E$ for fixed energy E. The additional linear term spoils the symmetry with respect to reflections in the variable u. In [44, 45] I have developed a procedure to deal with such problems within the path integral. We assume that we have evaluated a path integral problem with a potential $V(x)$ in, say, the entire $\mathbb{R}$. This path integral is called $K^{(V)}(T)$. The corresponding Green function is denoted by $G^{(V)}(E)$. Now we consider the path integral problem with the same potential V, but with Dirichlet (D) boundary conditions at the location $x = a$ and we consider the half-space $x > a$. Then the Green function in the half-space $x > a$ is given by [44, 45]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(x>a)}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a; E)G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E)}. \tag{22}$$

We identify the Green function $G^{(V)}(E)$ with (13) and obtain consequently

$$G^{(V_1)}(r'', r'; E) = 2(r'r'')^{1/4}\sqrt{\frac{m}{\pi\hbar^3\omega}}\Gamma(-\nu)$$
$$\times\left\{D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(\sqrt{r_>} - \frac{b}{2E}\right)\right]D_\nu\left[-\sqrt{\frac{2m\omega}{\hbar}}\left(\sqrt{r_<} - \frac{b}{2E}\right)\right]\right.$$
$$-D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(\sqrt{r'} - \frac{b}{2E}\right)\right]D_\nu\left[-\sqrt{\frac{2m\omega}{\hbar}}\left(\sqrt{r'} - \frac{b}{2E}\right)\right]$$
$$\left.\times D_\nu\left(\sqrt{\frac{2m\omega}{\hbar}}\frac{b}{2E}\right)\Bigg/ D_\nu\left(-\sqrt{\frac{2m\omega}{\hbar}}\frac{b}{2E}\right)\right\}. \tag{23}$$

This determines the energy spectrum by the zeros of the parabolic cylinder function, i.e.

$$D_{\nu_n}\left(-\sqrt{\frac{2m\omega_n}{\hbar}}\frac{b}{2E_n}\right) = 0. \tag{24}$$

This result is in accordance with [20]. I have indicated by $\nu_n = \nu(E_n)$ and $\omega_n = \omega(E_n)$ the explicit dependence on E_n. The analysis in [45] showed that the poles coming from the prefactor in (23) play no role in the corresponding boundary condition problem. The case $b = 0$ is contained in (23) by noting that for $b = 0$ the path integral (11) is a radial path integral in u with angular momentum $l = 0$ [5].

3.2. A radial confinement potential

The second potential we want to consider has the form ($r > 0$)

$$V_2(r) = \frac{m}{2}\omega^2 r^{2/3} + \frac{B}{r^{2/3}} + \frac{\hbar^2}{2m}\frac{\gamma}{r^2}. \tag{25}$$

This potential models a confinement potential for quark–antiquark interactions [21]; however it is not quite oscillator-like, and has a singularity at the origin. We perform a spacetime transformation as before with $r = u^{3/2}$. In doing that we set $\gamma = -\frac{5}{36}$ in order to cancel the corresponding term in the emerging quantum potential $\Delta V = 5\hbar^2/32mu^2$ due to the transformation. Hence we obtain

$$K^{(V_2)}(r'', r'; T) = \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \left(\frac{m}{2}\omega^2 r^{2/3} + \frac{B}{r^{2/3}} - \frac{\hbar^2}{2m}\frac{5}{36r^2}\right)\right]\mathrm{d}t\right\} \tag{26}$$
$$= \frac{3}{2}(r'r'')^{1/6}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\mathrm{e}^{-9\mathrm{i}Bs''/4\hbar}$$

$$\times \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{u}^2 + \frac{9}{4}Eu - \frac{9}{4}\frac{m}{2}\omega^2 u^2\right)\mathrm{d}s\right] \tag{27}$$

$$= \frac{3}{2}(r'r'')^{1/6}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\exp\left[\frac{\mathrm{i}}{\hbar}s''\frac{9}{4}\left(\frac{E^2}{2m\omega^2} - B\right)\right]$$
$$\times \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_0^{s''}(\dot{v}^2 - \tilde{\omega}^2 v^2)\,\mathrm{d}s\right]. \tag{28}$$

In the last step we have set $\tilde{\omega} = 3\omega/2$ and $v = u - E/m\omega^2$. Inserting the Green function for the harmonic oscillator gives the Green function for the path integral in the variable v ($v \in \mathbb{R}$):

$$G(v'', v'; E) = \sqrt{\frac{3m}{2\pi\hbar^3\omega}}\Gamma(-\nu)D_\nu\left(\sqrt{\frac{3m\omega}{\hbar}}v_>\right)D_\nu\left(-\sqrt{\frac{3m\omega}{\hbar}}v_<\right) \tag{29}$$

with $\nu = -\frac{1}{2} - \frac{3}{2}(E^2/2m\omega^2 - B)/\hbar\omega$. Provided $v \in \mathbb{R}$, the poles of the Green function would determine the energy spectrum given by $E_n = \sqrt{2m\omega^2[2(n+\frac{1}{2})\hbar\omega/3 + B]}$ for $n \in \mathbb{N}_0$ ($\omega^2 > 0$, $B > -\hbar\omega/3$) [19]. But for the same reasons as in section 3.1 this 'solution' must be discarded.

In order to obtain the proper solution in the half-space $v > -E/m\omega^2$ for fixed energy E, i.e. for the potential V_2, we proceed similarly as for the previous case and obtain

$$G^{(V_2)}(r'', r'; E) = \tfrac{3}{2}(r'r'')^{1/6}\sqrt{\frac{3m}{2\pi\hbar^3\omega}}\,\Gamma(-\nu)$$
$$\times\left\{D_\nu\left[\sqrt{\frac{3m\omega}{\hbar}}\left(r_>^{2/3} - \frac{E}{m\omega^2}\right)\right]D_\nu\left[-\sqrt{\frac{3m\omega}{\hbar}}\left(r_<^{2/3} - \frac{E}{m\omega^2}\right)\right]\right.$$
$$+ D_\nu\left[\sqrt{\frac{3m\omega}{\hbar}}\left(r'^{2/3} - \frac{E}{m\omega^2}\right)\right]D_\nu\left[\sqrt{\frac{3m\omega}{\hbar}}\left(r''^{2/3} - \frac{E}{m\omega^2}\right)\right]$$
$$\left.\times D_\nu\left(\sqrt{\frac{3m\omega}{\hbar}}\frac{E}{m\omega^2}\right)\Big/ D_\nu\left(-\sqrt{\frac{3m\omega}{\hbar}}\frac{E}{m\omega^2}\right)\right\}. \tag{30}$$

This determines the energy spectrum by the zeros of the parabolic cylinder function, i.e.

$$D_{\nu_n}\left(-\sqrt{\frac{3m\omega}{\hbar}}\frac{E_n}{m\omega^2}\right) = 0. \tag{31}$$

In the case that $\omega^2 < 0$ we have to replace $\omega \to \mathrm{i}\omega$ in (30). In the special case $\omega = 0$ the corresponding Green function can be obtained by using the Green function for the linear potential [45] and we get

$$G^{(V_2)}_{\omega=0}(r'', r'; E) = 2(r'r'')^{1/6}\frac{m}{\hbar^2}\left[\left(r'^{2/3} - \frac{B}{E}\right)\left(r''^{2/3} - \frac{B}{E}\right)\right]^{1/2}$$
$$\times\left\{K_{1/3}\left[\frac{\sqrt{-2mE}}{\hbar}\left(r_>^{2/3} - \frac{B}{E}\right)^{3/2}\right]I_{1/3}\left[\frac{\sqrt{-2mE}}{\hbar}\left(r_<^{2/3} - \frac{B}{E}\right)^{3/2}\right]\right.$$
$$-K_{1/3}\left[\frac{\sqrt{-2mE}}{\hbar}\left(r'^{2/3} - \frac{B}{E}\right)^{3/2}\right]K_{1/3}\left[\frac{\sqrt{-2mE}}{\hbar}\left(r''^{2/3} - \frac{B}{E}\right)^{3/2}\right]$$

$$\times I_{1/3}\left(-\frac{B}{\hbar E}\sqrt{2mB}\right)\Big/K_{1/3}\left(-\frac{B}{\hbar E}\sqrt{2mB}\right)\Bigg\}. \tag{32}$$

$I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions [40, p 958]. Possible bound states are determined by

$$K_{1/3}\left(-\frac{B}{\hbar E_n}\sqrt{2mB}\right)=0. \tag{33}$$

Due to the property

$$\mathrm{Ai}(z)=\frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2}K_{1/3}\left(\frac{2}{3}z^{3/2}\right) \tag{34}$$

the zeros are determined by the zeros of the Airy function $\mathrm{Ai}(z)$, $z=-\alpha_n$ ($\alpha_n>0, n\in\mathbb{N}_0$). From the relation

$$\mathrm{Ai}\left(-\frac{B}{E}\sqrt[3]{\frac{-9mE}{2\hbar^2}}\right)=0 \tag{35}$$

we see that negative bound states are allowed only if $B<0$. Hence it follows

$$E_n=-\sqrt{\frac{9m}{2\hbar^2}}\left(\frac{|B|}{\alpha_n}\right)^{3/2}\qquad n\in\mathbb{N}_0. \tag{36}$$

Therefore the singular term $B/r^{2/3}$ must be attractive in order that bound states can exist. Note that no resonance states exist because all zeros of $\mathrm{Ai}(z)$ are located on the negative real axis [46, p 166]. The ground-state energy is $E_0=-\sqrt{9m/2\hbar^2}(|B|/2.341\ldots)^{3/2}$, and the accumulation point is $E_\infty=0$.

3.3. *A modified Rosen–Morse potential I*

The third potential I want to consider has the form ($x\in\mathbb{R}$)

$$V_3(x)=-\frac{A}{\sqrt{1+\mathrm{e}^{-2x}}}+\frac{B}{1+\mathrm{e}^{-2x}}+\frac{C}{(1+\mathrm{e}^{-2x})^2}. \tag{37}$$

I perform the transformation $x=\ln(\sinh u)$, $u>0$, together with the appropriate time transformation. The emerging quantum potential is

$$\Delta V(u)=-\frac{\hbar^2}{8m}\left(\frac{3}{\cosh^2 u}+\frac{1}{\sinh^2 u}\right). \tag{38}$$

In order that the terms $\propto -1/\cosh^2 u$ cancel we must set $C=-3\hbar^2/8m$. This gives for the path integral

$$\begin{aligned}
K^{(V_3)}(r'',r';T) &= \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-\left(\frac{B}{1+\mathrm{e}^{-2x}}-\frac{A}{\sqrt{1+\mathrm{e}^{-2x}}}-\frac{3\hbar^2}{8m(1+\mathrm{e}^{-2x})^2}\right)\right]\mathrm{d}t\right\}\\
&= (\coth u'\coth u'')^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{3\hbar^2}{8m}-B+E\right)s''\right]\\
&\quad\times\int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{u}^2+A\coth u+\hbar^2\frac{2mE/\hbar^2+1/4}{2m\sinh^2 u}\right)\mathrm{d}s\right].
\end{aligned} \tag{39}$$

The path integral in the variable u is the path integral for the Manning–Rosen potential $V_{\mathrm{MR}}(u) = -A\coth u + B'/\sinh^2 u$ [47]. From the spectral representation of the Manning–Rosen potential we therefore derive the following quantization condition for the bound states ($n = 0, 1, \ldots, N_{\max}$):

$$B - \frac{3\hbar^2}{8m} - E_n = \frac{\hbar^2}{2m}(n + \tfrac{1}{2} + \sqrt{-2mE_n}/\hbar)^2 + \frac{mA^2/2}{(n + \frac{1}{2} + \sqrt{-2mE_n}/\hbar)^2}. \tag{40}$$

This can be rewritten into a cubic equation for $\sqrt{-E_n}$:

$$\begin{aligned} &\frac{2(n+\frac{1}{2})}{\hbar}\sqrt{2m}(-E_n)^{3/2} + \left[5(n+\tfrac{1}{2})^2 - \frac{2m}{\hbar^2}\left(B - \frac{3\hbar^2}{8m}\right)\right](-E_n) \\ &\qquad + 2\left[\frac{\hbar^2(n+\frac{1}{2})^2}{m} - \left(B - \frac{3\hbar^2}{8m}\right)\right](n+\tfrac{1}{2})\frac{\sqrt{2m}}{\hbar}(-E_n)^{1/2} \\ &\qquad + \left[\frac{\hbar^2}{2m}(n+\tfrac{1}{2})^4 + \frac{mA^2}{2} - \left(B - \frac{3\hbar^2}{8m}\right)\right] = 0. \end{aligned} \tag{41}$$

On the other hand we obtain from the Green function representation of the Manning–Rosen potential [11, 28, 48]

$$\begin{aligned} G^{(V_3)}(x'', x'; E) &= (\coth u' \coth u'')^{1/2}\frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_{E'})\Gamma(L_{E'} + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \\ &\quad \times \left(\frac{1}{1+u'}\frac{1}{1+u''}\right)^{\frac{1}{2}(m_1+m_2+1)}\left(\frac{u'}{1+u'}\frac{u''}{1+u''}\right)^{\frac{1}{2}(m_1-m_2)} \\ &\quad \times {}_2F_1\left(-L_{E'} + m_1, L_{E'} + m_1 + 1; m_1 - m_2 + 1; \frac{u_<}{1+u_<}\right) \\ &\quad \times {}_2F_1\left(-L_{E'} + m_1, L_{E'} + m_1 + 1; m_1 + m_2 + 1; \frac{1}{1+u_>}\right). \end{aligned} \tag{42}$$

Here $\sinh u = \mathrm{e}^x$, $L_{E'} = -\frac{1}{2} + \frac{1}{2}\sqrt{2m(A - E')}/\hbar$, $m_{1/2} = \frac{1}{2}(s \pm \sqrt{-2m(A + E')}/\hbar)$, $s = 2\sqrt{-2mE}/\hbar$, $E' = 3\hbar^2/8m - B + E$, and ${}_2F_1(a, b; c; z)$ is the hypergeometric function [40, p 1039]. The poles of the Green function determine the energy spectrum which coincidences with the one determined in (40), and the corresponding residua give the wavefunction expansion ($s_n = s(E_n)$, $k_1 = \frac{1}{2}[1 + \frac{1}{2}(s_n + 2n + 1) + 2mA/\hbar^2(s_n + 2n + 1)]$)

$$\begin{aligned} \Psi_n^{(V_3)}(x) &= N_{E_n}^{(V_3)}\sqrt{1 + \frac{4mA}{\hbar^2(s_n + 2n + 1)^2}} \\ &\quad \times \left[\frac{(2k_1 - n - s_n - 2)n!\,\Gamma(2k_1 - n - 1)}{\Gamma(n + s_n + 1)\Gamma(2k_1 - s_n - n - 1)}\right]^{1/2} \\ &\quad \times \sqrt{\coth u}\,(1 - \mathrm{e}^{-2u})^{(s_n+1)/2}\mathrm{e}^{-2u(k_1 - s_n/2 - n - 1)} \\ &\quad \times P_n^{(2k_1 - 2n - s_n - 2, s_n)}(1 - 2\mathrm{e}^{-2u}) \end{aligned} \tag{43}$$

$$E_n = -\frac{\hbar^2}{2m}\left(\sqrt[3]{\sqrt{D} + \frac{Q}{2}} - \sqrt[3]{\sqrt{D} - \frac{Q}{2}} + \frac{R}{3}\right)^2 \tag{44}$$

$$R = \frac{5(n+\frac{1}{2})^2 - 2m(B - 3\hbar^2/8m)/\hbar^2}{2(n+\frac{1}{2})} \tag{45}$$

$$S = 2\left[(n+\tfrac{1}{2})^2 - m/\hbar^2\left(B - 3\hbar^2/8m\right)\right] \tag{46}$$

$$T = m\frac{\hbar^2/2m(n+\frac{1}{2})^4 + mA^2/2 - (B - 3\hbar^2/8m)}{(n+\frac{1}{2})\hbar^2} \tag{47}$$

$$N_{E_n}^{(V_3)} = \left(\frac{\sqrt{-2mE_n}\,(n+\frac{1}{2}+\sqrt{-2mE_n}/\hbar)^2}{\hbar(n+\frac{1}{2})[3(\sqrt{-2mE_n}/\hbar + R/3)^2 - R^2/3 + S]}\right)^{1/2}. \tag{48}$$

The quantities D and Q are defined analogously as in (17). In order for a potential well and bound states to exist we see from (37) that A and B should be positive with $A < 2B$ (for simplicity I have set $C = 0$). The maximal number $N_{\max}$ of bound states is found by requiring $E_n < 0$. The cut of the Green function determines the continuous spectrum, and the corresponding wavefunctions are determined by the method described in [48]. Thus the wavefunctions and the energy spectrum of the continuous states are given by

$$\Psi_p^{(V_3)}(x) = N_p^{(V_3)}\sqrt{\coth u}\, u^{-\mathrm{i}p/2}(u-1)^{[\mathrm{i}p-(1+s_p)]/2} \times {}_2F_1\left(\frac{1+s_p+\mathrm{i}(\tilde{p}-p)}{2}, \frac{1+s_p-\mathrm{i}(\tilde{p}+p)}{2}; s_p+1; \frac{1}{1-u}\right) \tag{49}$$

$$N_p^{(V_3)} = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa) \times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)]^{1/2} \tag{50}$$

[$\kappa = \frac{1}{2}(1+\mathrm{i}p), k_2 = \frac{1}{2}(1+s_p), s_p = s(E_p), \tilde{p} = \sqrt{2m(E_p - A)}/\hbar, p \in \mathbb{R}$] with energy spectrum $E_p = \hbar^2p^2/2m - A + B - 3\hbar^2/8m$. The results concerning the bound states coincide with [18]. Note that the transformation $x = \frac{1}{2}\ln(-\cosh^2 v)$ leads to a Rosen-Morse potential. However, this transformation is no longer a real transformation which causes interpretation difficulties, and therefore it is not used.

3.4. A modified Rosen–Morse potential II

The fourth potential I treat has the form ($x \in \mathbb{R}$)

$$V_4(x) = \frac{A}{1+\mathrm{e}^{-2x}} - \frac{B\mathrm{e}^{-x}}{\sqrt{1+\mathrm{e}^{-2x}}} - \frac{3\hbar^2}{8m(1+\mathrm{e}^{-2x})^2}. \tag{51}$$

I perform the same transformation $x = \ln(\sinh u)$ together with the appropriate time transformation, and I have set the coupling in the third term equal to $C = -3\hbar^2/8m$ in order that it cancels with the corresponding term in the emerging quantum potential. This gives for the path integral

$$\begin{aligned} K^{(V_4)}(r'', r'; T) &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \\ &\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \left(\frac{A}{1+\mathrm{e}^{-2x}} - \frac{B\mathrm{e}^{-x}}{\sqrt{1+\mathrm{e}^{-2x}}} - \frac{3\hbar^2}{8m(1+\mathrm{e}^{-2x})^2}\right)\right]\mathrm{d}t\right\} \\ &= (\coth u' \coth u'')^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{\hbar^2}{8m} - A\right)s''\right] \\ &\quad\times \int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{u}^2 + B\frac{\coth u}{\sinh u} + \hbar^2\frac{2mE/\hbar^2 + 1/4}{2m\tanh^2 u}\right)\mathrm{d}s\right]. \end{aligned} \tag{52}$$

The path integral in the variable u is a path integral of a hyperbolic Scarf-like potential

$$V_{\mathrm{HSL}}(u)=\frac{\hbar^2}{2m}\left(V_0+V_1\coth^2 u+V_2\frac{\coth u}{\sinh u}\right)$$

as discussed in [28]. We identify $V_0=2mA/\hbar^2-\frac{1}{2}$, $V_1=-(2mE/\hbar^2+\frac{1}{4})$, $V_2=-2mB/\hbar^2$. It has energy eigenvalues

$$E_n=\frac{\hbar^2}{2m}(V_0+V_1)-\frac{\hbar^2}{8m}[2(k_1-k_2-n)-1)]^2.$$

From its spectral expansion we derive the quantization condition for the potential V_4 ($n=0,1,\dots,N_{\max}$):

$$\sqrt{A-E_n-\frac{3\hbar^2}{8m}}=\frac{1}{2}(\sqrt{B-E_n}-\sqrt{-B-E_n})-\frac{\hbar}{\sqrt{2m}}(n+\tfrac{1}{2}) \tag{53}$$

This gives, after some algebra, a cubic equation in $(-E_n)$ ($\lambda=A+C+\tilde{n}^2, C=-3\hbar^2/8m, \tilde{n}=\hbar(n+\frac{1}{2})/\sqrt{2m}$)

$$\begin{aligned}4\tilde{n}^2(-E_n)^3&+[12\tilde{n}^2(\tilde{n}^2+\lambda)-\lambda^2](-E_n)^2\\&+\left[16\tilde{n}^2\lambda(A+C+\lambda)-2(\lambda+4\tilde{n}^2)\left(\lambda^2+\frac{B^2}{4}+4\tilde{n}^2(A+C)\right)\right](-E_n)\\&+\left[16\tilde{n}^2\lambda^2(A+C)-\left(\lambda^2+\frac{B^2}{4}+4\tilde{n}^2(A+C)\right)^2\right]=0.\end{aligned} \tag{54}$$

From the Green function of the hyperbolic Scarf-like potential we derive the Green function for the potential V_4:

$$\begin{aligned}G^{(V_4)}(x'',x';E)&=\frac{2m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\&\times(\cosh u'\cosh u'')^{-(m_1-m_2)}(\tanh u'\tanh u'')^{m_1+m_2+1}\\&\times{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 u_<}\right)\\&\times{}_2F_1(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh^2 u_>)\end{aligned} \tag{55}$$

with $\sinh u = e^x$, $m_{1,2}=\eta/2\pm\sqrt{V_0+V_1-8mE/\hbar^2}$, and where $\eta=\sqrt{V_1+V_2+1/4}$, $L_\nu=\frac{1}{2}(\nu-1)$, $\nu=\sqrt{V_1-V_2+1/4}$. Again, the poles of the Green function determine the energy spectrum which coincidences with the one determined in (53), and the corresponding residua give the wavefunction expansions. We obtain $\left(k_1=\frac{1}{2}(1+\nu), k_2=\frac{1}{2}(1+\eta), \eta=\sqrt{-2m(E_n+B)}\big/\hbar, \nu=\sqrt{2m(B-E_n)}\big/\hbar\right)$

$$E_n=\sqrt[3]{\sqrt{D}+\frac{Q}{2}}-\sqrt[3]{\sqrt{D}-\frac{Q}{2}}+\frac{R}{3} \tag{56}$$

$$R=\frac{12\tilde{n}^2(\tilde{n}^2+\lambda)-\lambda^2}{4\tilde{n}^2} \tag{57}$$

$$T=\frac{16\tilde{n}^2\lambda^2(A+C)-[\lambda^2+B^2/4+4\tilde{n}^2(A+C)]^2}{4\tilde{n}^2} \tag{58}$$

$$S=\frac{8\tilde{n}^2\lambda(A+C+\lambda)-(\lambda^2+4\tilde{n}^2)(\lambda^2+B^2/4+4\tilde{n}^2(A+C)]}{2\tilde{n}^2} \tag{59}$$

$$\Psi_n^{(V_4)}(x)=N_{E_n}^{(V_4)}\left[\frac{(2k_1-2k_2-2n-1)n!\,\Gamma(2k_1-n-1)}{2\Gamma(2k_2+n)\Gamma(2k_1-2k_2-n)}\right]^{1/2}$$

$$\times\sqrt{\coth u}\left(\sinh\frac{u}{2}\right)^{2k_2-1/2}\left(\cosh\frac{u}{2}\right)^{2n-2k_1+3/2}$$
$$\times P_n^{[2k_2-1,2(k_1-k_2-n)-1]}\left(\frac{2}{\cosh^2\frac{u}{2}}-1\right) \tag{60}$$

$$N_{E_n}^{(V_4)}=\left(\frac{(2\lambda-E_n)\sqrt{E_n^2-B^2}\,(\sqrt{A-E_n+C}+\tilde{n}}{\tilde{n}[3(R/3-E_n)^2-R^2/3+S]}\right)^{1/2}. \tag{61}$$

The quantities D and Q are defined analogously as in (17). A potential well and bound states exist if $A < 0, 0 < B < |A|$ [18], and the number $N_{\max}$ of bound states is found by requiring $|E_n| > B$. The scattering states have the form [$\kappa = \frac{1}{2}(1 + \mathrm{i}p)$, $p \in \mathbb{R}$]

$$\Psi_p^{(V_4)}(u)=N_p^{(V_4)}\sqrt{\coth u}\left(\cosh\frac{u}{2}\right)^{2n-2k_1+3/2}\left(\sinh\frac{u}{2}\right)^{2k_2-1/2}$$
$$\times{}_2F_1\left(k_1+k_2-\kappa,k_2-k_1-\kappa+1;2k_2;\tanh^2\frac{u}{2}\right) \tag{62}$$

$$N_p^{(V_4)}=\frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh 2\pi p}{2\pi^2}}[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)$$
$$\times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)]^{1/2}. \tag{63}$$

The results concerning the bound states coincide with [18].

4. Summary

In this article the path integral treatments of four so-called 'conditionally exactly solvable' potentials have been presented. Our approach showed that the path integral in the present cases is far superior in comparison to other methods. In spite of the fact that the bound-state energy levels could not be stated in closed form in the first two cases, closed form solutions in terms of the Green function were still possible. The poles of the Green functions (transcendental equations in terms of parabolic cylinder functions) gave the bound-state energy levels, the cuts provided the scattering states.

In the second set of potentials the bound-state solutions are determined by a cubic equation which considerably complicated the expressions analytically. In each of the two cases the bound-state energy levels with the wavefunctions and the scattering solutions could be obtained.

The results are rather satisfactory. In the Schrödinger approach, be it the usual study in non-relativistic quantum mechanics or a super-symmetric investigation, the potential problem is not seen as a whole. In comparison, the path integral provides comprehensive information, about the propagator, when it can be explicitly computed, the Green function with its poles and cuts, the bound-state wavefunctions, the continuous spectrum and the necessary boundary conditions. We also see that the interplay of various techniques was needed to obtain the proper solutions. In all four cases a spacetime transformation was essential. In the first two cases, its was not only necessary to know about the path integral solution of a (shifted) harmonic oscillator or the linear potential, but it was even more essential to know how to incorporate explicit boundary conditions into the path integral. In the two modified Rosen–Morse oscillators two special path integral solutions had to be known which are in turn based on the path integral solution of the modified Pöschl–Teller potential.

It is to be expected that in the future some other specific path integral solutions can be found by relating known problems to more complicated (and therefore more realistic) potentials, which can incorporate more parameters. In fact, it is possible to modify the Natanzon potentials [49] in such a way that the four conditionally solvable potentials, which have been discussed here, are a two-parameter subclass of a class of actually four-parametric potentials which may be called 'conditionally solvable Natanzon potentials'. These considerations can, of course, also be extended to two and three dimensions.

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